27. Vector fields in space

A vector field in space is given by

$$\vec{F} = P\hat{\imath} + Q\hat{\jmath} + R\hat{k} = \langle P, Q, R\rangle.$$

Here the components, P, Q and R are scalar functions of x, y and z.

$\vec{F}$ could be a force field;

$$\vec{F} = -\frac{c\langle x, y, z\rangle}{\rho^3},$$

is the force due to gravity. There is both an electric $\vec{E}$ and a magnetic field $\vec{B}$. There are velocity fields $\vec{v}$ and gradient vector fields.

In space, we can measure the flux of $\vec{F}$ across a surface S,

$$\iint_S \vec{F} \cdot \hat{n}\, \mathrm{d}S.$$

Here $\hat{n}$ is a unit normal to the surface. There are two choices of $\hat{n}$; we have to choose an orientation, a direction which we decide is positive.

Notation:

$$\mathrm{d}\vec{S} = \hat{n}\, \mathrm{d}S.$$

Suppose that $\vec{F}$ represents the velocity vector field of some fluid. The amount of water that crosses a small piece of surface in unit time is approximately a parallelepiped with area of base ΔS and height $\vec{F} \cdot \hat{n}$,

$$\vec{F} \cdot \hat{n}\Delta S.$$

Suppose

$$\vec{F} = x\hat{\imath} + y\hat{\jmath} + z\hat{k},$$

and S is the surface of a sphere of radius a, centred at the origin. Orient the surface S so that the unit normal points outwards,

$$\hat{n} = \frac{1}{a}\langle x, y, z\rangle.$$

In this case

$$\vec{F} \cdot \hat{n} = \frac{1}{a}(x^2 + y^2 + z^2) = a.$$

Hence

$$\iint_S \vec{F} \cdot \hat{n}\, \mathrm{d}S = \iint_S a\, \mathrm{d}S = 4\pi a^3.$$

Now suppose we work with $\vec{F} = z\hat{k}$. Then

$$\vec{F} \cdot \hat{n} = \frac{z^2}{a}.$$

So the flux is

$$\iint_S \vec{F}\cdot\hat{n}\,\mathrm{d}S = \iint_S \frac{z^2}{a}\,\mathrm{d}S = \int_0^{2\pi}\int_0^{\pi} \frac{a^2\cos^2\phi}{a} a^2 \sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta.$$

The inner integral is

$$\int_0^{\pi} a^3\cos^2\phi\sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta = \left[-\frac{a^3}{3}\cos^3\phi\right]_0^{\pi} = \frac{2a^3}{3}.$$

The outer integral is

$$\int_0^{2\pi} \frac{2a^3}{3}\,\mathrm{d}\theta = \frac{4\pi a^3}{3}.$$

In general, it can be quite hard to parametrise a surface. We will need two parameters to describe the surface and we must express

$$\vec{F}\cdot\hat{n}\,\mathrm{d}S,$$

in terms of them. We must also orient the surface:

Question 27.1. *Can one always orient a surface?*

In fact, somewhat surprisingly, the answer is no. The Möbius band is a surface that cannot be oriented.

To begin with, here are some easy special cases:

(1) If $z = a$ is a horizontal plane then

$$\mathrm{d}\vec{S} = \hat{k}\,\mathrm{d}x\,\mathrm{d}y,$$

(here we choose the upwards orientation).

(2) For the surface of a sphere of radius a centred at the origin then

$$\mathrm{d}\vec{S} = \hat{n}a^2\sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta,$$

where

$$\hat{n} = \frac{1}{a}\langle x, y, z\rangle,$$

so that

$$\mathrm{d}\vec{S} = a\sin\phi\langle x, y, z\rangle\,\mathrm{d}\phi\,\mathrm{d}\theta.$$

(3) For a cylinder of radius a centred on the z-axis, use z, θ.

$$\hat{n} = \frac{1}{a}\langle x, y, 0\rangle,$$

which points radially out of the cylinder.

$$\mathrm{d}S = a\,\mathrm{d}z\,\mathrm{d}\theta,$$

so that

$$\mathrm{d}\vec{S} = \langle x, y, 0\rangle\,\mathrm{d}z\,\mathrm{d}\theta.$$

(4) For the graph of a function $f(x, y)$,

$$\mathrm{d}\vec{S} = \langle -f_x, -f_y, 1 \rangle \mathrm{d}x \, \mathrm{d}y.$$