30. Line integrals

If

$$\vec{F} = P\hat{\imath} + Q\hat{\jmath} + R\hat{k},$$

is a vector field on $\mathbb{R}^3$ and C is a curve in space then we can define the line integral

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r},$$

in the same way as we did in the plane. If we pick a parametrisation of C,

$$\vec{r}(t) = \langle x(t), y(t), z(t)\rangle, \qquad \text{and} \qquad a \leq t \leq b$$

then we can express

$$\mathrm{d}\vec{r} = \langle \mathrm{d}x, \mathrm{d}y, \mathrm{d}z\rangle = \vec{v}(t)\,\mathrm{d}t \qquad \text{and} \qquad \vec{F} = \langle P, Q, R\rangle$$

in terms of t and integrate this. If $\vec{F}$ represents force, the line integral represents the work done moving a particle from the start P_0 to the end P_1.

Example 30.1. *Let C be the parametric curve*

$$\vec{r}(t) = \langle t, t^2, t^3\rangle \qquad 0 \leq t \leq 1 \qquad \textit{and} \qquad \vec{F} = \langle yz, xz, xy\rangle.$$

C is called the twisted cubic.

We express everything in terms of t,

$$\mathrm{d}\vec{r} = \langle 1, 2t, 3t^2\rangle\,\mathrm{d}t \qquad \text{and} \qquad \vec{F} = \langle t^5, t^4, t^3\rangle.$$

So

$$\vec{F} \cdot \mathrm{d}\vec{r} = \langle t^5, t^4, t^3\rangle \cdot \langle 1, 2t, 3t^2\rangle\,\mathrm{d}t = 6t^5\,\mathrm{d}t.$$

The work done is then

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r} = \int_0^1 6t^5\,\mathrm{d}t = \Big[t^6\Big]_0^1 = 1.$$

Now let's suppose we go from $P_0 = (0,0,0)$ to $P_1 = (1,1,1)$ along a different path. Let's say we go parallel to $\hat{\imath}$, C_1, parallel to $\hat{\jmath}$, C_2 and then parallel to $\hat{k}$, C_3. Let

$$C' = C_1 + C_2 + C_3.$$

So

$$\int_{C'} \vec{F} \cdot \mathrm{d}\vec{r} = \int_{C_1} \vec{F} \cdot \mathrm{d}\vec{r} + \int_{C_2} \vec{F} \cdot \mathrm{d}\vec{r} + \int_{C_3} \vec{F} \cdot \mathrm{d}\vec{r}.$$

Parametrise C_1 in the obvious way

$$\vec{r}(t) = \langle t, 0, 0\rangle \qquad 0 \leq t \leq 1.$$

Then

$$\mathrm{d}\vec{r} = \langle 1,0,0\rangle\,\mathrm{d}t \qquad \text{and} \qquad \vec{F} = \langle 0,0,0\rangle.$$

So

$$\int_{C_1} \vec{F}\cdot \mathrm{d}\vec{r} = 0.$$

Parametrise C_2

$$\vec{r}(t) = \langle 1,t,0\rangle \qquad 0\leq t\leq 1.$$

Then

$$\mathrm{d}\vec{r} = \langle 0,1,0\rangle\,\mathrm{d}t \qquad \text{and} \qquad \vec{F} = \langle 0,0,t\rangle.$$

We have

$$\vec{F}\cdot \mathrm{d}\vec{r} = \langle 0,0,1\rangle\cdot\langle 0,1,0\rangle\,\mathrm{d}t = 0\,\mathrm{d}t.$$

So

$$\int_{C_2} \vec{F}\cdot \mathrm{d}\vec{r} = 0.$$

Parametrise C_3

$$\vec{r}(t) = \langle 1,1,t\rangle \qquad 0\leq t\leq 1.$$

Then

$$\mathrm{d}\vec{r} = \langle 0,0,1\rangle\,\mathrm{d}t \qquad \text{and} \qquad \vec{F} = \langle t,t,1\rangle.$$

We have

$$\vec{F}\cdot \mathrm{d}\vec{r} = \langle t,t,1\rangle\cdot\langle 0,0,1\rangle\,\mathrm{d}t = \mathrm{d}t.$$

So

$$\int_{C_3} \vec{F}\cdot \mathrm{d}\vec{r} = \int_0^1 \mathrm{d}t = 1.$$

The reason why both answers are the same is that $\vec{F}$ is a gradient vector field,

$$\vec{F} = \langle yz,xz,xy\rangle = \nabla(xyz) = \nabla f,$$

where $f(x,y,z) = xyz$. As before we have the fundamental theorem of calculus for line integrals

$$\int_C \nabla f\cdot \mathrm{d}\vec{r} = f(P_1) - f(P_0).$$

As before this means the integral is path independent and $\vec{F}$ is conservative. In our case

$$f(0,0,0) = 0 \qquad f(1,1,1) = 1, \qquad \text{so that} \qquad f(1,1,1) - f(0,0,0) = 1.$$

Theorem 30.2. *Let $\vec{F}$ be a vector field on $\mathbb{R}^3$ (or more generally a simply connected region; more about this later).*

$\vec{F}$ is a gradient vector field if and only if

$$P_y = Q_x, \qquad P_z = R_x \qquad \textit{and} \qquad Q_z = R_y.$$

Again, one direction is reasonably clear. If

$$\vec{F} = \nabla f,$$

then

$$P = f_x, \qquad Q = f_y \qquad \text{and} \qquad R = f_z,$$

so that

$$P_y = f_{xy} \qquad \text{and} \qquad Q_x = f_{yx},$$

so that the equality $P_y = Q_x$ is just saying the mixed partials of f are equal.

Example 30.3. *For which a and b is*

$$\vec{F} = axy\hat{\imath} + (x^2 + z^3)\hat{\jmath} + (byz^2 - 4z^3)\hat{k}$$

a gradient vector field?

We have

$$P = axy, \qquad Q = x^2 + z^3 \qquad \text{and} \qquad R = byz^2 - 4z^3.$$

$Q_x = 2x$ and $P_y = ax$, so that

$$ax = P_y = Q_x = 2x,$$

and so $a = 2$. $Q_z = 3z^2$ and $R_y = bz^2$ so that

$$bz^2 = R_y = Q_z = 3z^2$$

and so $b = 3$. Finally,

$$P_z = 0 \qquad \text{and} \qquad R_x = 0,$$

so $P_z = R_x$ is clear. Hence

$$\vec{F} = 2xy\hat{\imath} + (x^2 + z^3)\hat{\jmath} + (3yz^2 - 4z^3)\hat{k}$$

is conservative.

Let's look for a potential function $f(x, y, z)$. We want to solve three PDEs

$$\frac{\partial f}{\partial x} = 2xy, \qquad \frac{\partial f}{\partial y} = x^2 + z^3 \qquad \text{and} \qquad \frac{\partial f}{\partial z} = 3yz^2 - 4z^3.$$

We have

$$\frac{\partial f}{\partial x} = 2xy.$$

Integrate both sides with respect to x.

$$f(x, y, z) = \int 2xy \, \mathrm{d}x = x^2y + g(y, z).$$

Note that the constant of integration is actually a function of both y and z. In other words
$$\frac{\partial g(y,z)}{\partial x} = 0.$$
Let's take this answer for f and plug this into the second PDE.
$$\frac{\partial f}{\partial y} = x^2 + z^3 \qquad \text{so that} \qquad x^2 + \frac{\partial g(y,z)}{\partial z} = x^2 + z^3.$$
Cancelling we get
$$\frac{\partial g(y,z)}{\partial y} = z^3.$$
Integrating this with respect to y we get
$$g(y,z) = \int z^3 \, \mathrm{d}y = z^3 y + h(z),$$
where the constant of integration is an abritrary function of z. So now we know
$$f(x,y,z) = x^2 y + z^3 y + h(z).$$
Finally, we plug this back into the third PDE:
$$\frac{\partial f}{\partial z} = 3yz^2 - 4z^3 \qquad \text{so that} \qquad 3yz^2 + \frac{dh}{dz} = 3yz^2 - 4z^3.$$
Cancelling we get
$$\frac{dh}{dz} = -4z^3.$$
Integrating with respect to z, we get
$$h(z) = -z^4 + c.$$
We can take $c = 0$. Putting all of this together
$$f(x,y,z) = x^2 y + z^3 y - z^4.$$

One can define a vector field which measures how far $\vec{F}$ is from being conservative.
$$\begin{aligned}
\operatorname{curl} \vec{F} &= \nabla \times \vec{F} \\
&= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \\
&= \hat{\imath} \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{vmatrix} - \hat{\jmath} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{vmatrix} + \hat{k} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix} \\
&= (R_y - Q_z)\hat{\imath} - (R_x - P_z)\hat{\jmath} + (Q_x - P_y)\hat{k}.
\end{aligned}$$

This is the curl of the vector field $\vec{F}$. It measures the angular velocity. For example,

$$\vec{v} = \langle -\omega y, \omega x, 0 \rangle,$$

represents rotation around z-axis with constant angular velocity ω.

$$\operatorname{curl} \vec{v} = \nabla \times \vec{v} = 2\omega \hat{k}.$$

So the magnitude is twice the angular velocity and the direction is the axis of rotation.