9. Max-min problems

Let $z = f(x, y)$ be a differentiatiable function. If f has a local minimum or maximum at (x_0, y_0) then $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0)$ are both zero. In this case the tangent plane is horizontal, since the tangent plane is determined by the two tangent lines to the coordinate cross-sections, and these are both horizontal. Equivalently, by the approximation formula, $\Delta f \approx 0$ (or even better, $|\Delta f|$ is much smaller than either $|\Delta x$ or $|\Delta y|$).

We say that (x_0, y_0) is a critical point of $f(x, y)$ if

$$f_x(x_0, y_0) = 0 \qquad \text{and} \qquad f_y(x_0, y_0) = 0.$$

Critical points come in three different flavours: a local minimum, a local maximum or a saddle point.

Question 9.1. *Find the critical points of*

$$f(x, y) = 5x^2 + 2xy + 10y^2 + 18x - 16y + 30.$$

The partials are

$$f_x(x, y) = 10x + 2y + 18 \qquad \text{and} \qquad f_y(x, y) = 2x + 20y - 16.$$

If we set those equal to zero, we get the critical points. We get two simultaneous linear equations,

$$5x + y = -9$$
$$x + 10y = 8.$$

Hence $49y = 49$, $y = 1$ and so $x = -2$. $(-2, 1)$ is the only critical point. Note that

$$f(x, y) = (x + 3y - 1)^2 + (2x - y + 5)^2 + 4,$$

and so we have a minimum. The minimum value is 4. Note that we have a paraboloid (shifted and rescaled).

Suppose that we want to maximise and minimise a function of two variables. If we have a function of one variable on an interval we find the critical points, check to see if they are maxima or minima (possibly using the 2nd derivative test) and then compare these with the values at the endpoints.

For a function of two variables we will do the same thing but note that checking to see what happens at the endpoints now means checking to see what happens on the boundary of the region.

Question 9.2. *What is the maximum and minimum value of*

$$f(x, y) = 5x^2 + 2xy + 10y^2 + 18x - 16y + 30.$$

over the region $-3 \le x \le 3$ *and* $-2 \le y \le 2$?

The region is a rectangle with sides $x = 3$, $-2 \le y \le 2$; $x = -33$, $-2 \le y \le 2$; $-3 \le x \le 3$, $y = 2$; $-3 \le x \le 3$, $y = -2$:
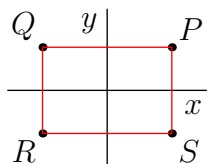

FIGURE 1. Boundary of region in xy-plane

Since there is only one critical point, a local minimum inside the boundary, we know the minimum is 4 (think about why this is true!), but the maximum must occur on the boundary. When $y = 2$ we get

$$g(x) = f(x, 2) = 5x^2 + 4x + 40 + 18x - 32 + 30 = 5x^2 + 22x - 38,$$

a function of one variable. $g(x)$ is a parabola, so has a local minimum. The maximum of $g(x)$ is at one of the endpoints, $x = 3$ or $x = -3$, which correspond to $P = (3, 2)$ and $Q = (-3, 2)$ for $f(x, y)$. A similar analysis along the other three edges tells us the maximum must occur at one of P, Q, R and S. Plugging in values we see that the maximum is at $R = (3, -2)$ and the maximum value is 189.

Suppose we have a bunch of data, a collection of pairs (x_i, y_i), $1 \le i \le n$. We try to fit a curve to this data. Quite often we expect the data to lie along a line. We posit that the data fits the equation,

$$y = ax + b.$$

Here a and b are unknowns and we try to find the best choice of a and b. The deviations are

$$y_i - (ax_i + b).$$

We choose a and b to minimise the sum of the squares of the deviations

$$D = \sum_i (y_i - (ax_i + b))^2.$$

We find the critical points of D,

$$\frac{\partial D}{\partial a} = 0 \qquad \text{and} \qquad \frac{\partial D}{\partial b} = 0.$$

This gives two equations

$$\sum_i 2(y_i - (ax_i + b))(-x_i) = 0 \qquad \text{and} \qquad \sum_i 2(y_i - (ax_i + b))(-1) = 0.$$

Collecting together terms in these expressions we get

$$(\sum_i x_i^2)a + (\sum_i x_i)b = \sum_i x_i y_i$$

$$(\sum_i x_i)a + nb = \sum_i y_i.$$

The same method works when we try to fit data to other functions. For example, imagine trying to fit data to exponential functions $y = ce^{ax}$. In this case take logs of both sides to get

$$\ln y = \ln(ce^{ax}) = \ln c + ax.$$

Put $b = \ln c$ and we are down to the method of least squares, with the data $(x_i, \ln y_i)$.

For example, Moore's law predicts that the number of transistors on an integrated circuit doubles roughly every two years. The following picture (source: wikipedia) plots the year against a log of the number of the transitors. Note the almost uncanny fit.

Microprocessor Transistor Counts 1971-2011 & Moore's Law

Transistor count (y-axis)

- 2,600,000,000
- 1,000,000,000
- 100,000,000
- 10,000,000
- 1,000,000
- 100,000
- 10,000
- 2,300

16-Core SPARC T3

Six-Core Core i7
Six-Core Xeon 7400
● 10-Core Xeon Westmere-EX

Dual-Core Itanium 2 ●
8-core POWER7
AMD K10
Quad-core z196
Quad-Core Itanium Tukwila
POWER6 ●
8-Core Xeon Nehalem-EX
Itanium 2 with 9MB cache ●
Six-Core Opteron 2400
AMD K10 ●
Core i7 (Quad)
Core 2 Duo
Itanium 2 ●
Cell

● AMD K8

● Barton
Pentium 4 ●
● Atom

AMD K7
AMD K6-III
AMD K6
Pentium III
● Pentium II

● AMD K5
80486 ●
Pentium

curve shows transistor
count doubling every
two years

80386 ●

80286 ●

68000 ●
● 80186

8086 ● ● 8088

8085
6800
● 6809
8080
● Z80

8008 ●
● MOS 6502

4004 ●
RCA 1802

1971   1980   1990   2000   2011

Date of introduction

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